



Logarithmic Harnack inequalities for general graphs with positive Ricci curvature



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ABSTRACT

In this paper, we prove Logarithmic Harnack inequalities for general graphs with positive Ricci curvature. As a consequence, we derive lower estimates for the log-Sobolev constant of graph G , extending previous results of Chung and Yau for Ricci flat graphs.

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1. Introduction

Harnack inequality and Logarithmic Harnack inequality play a very important role in many fields, such as [6,8], and in the past twenty years, these two inequalities have been established on graphs.

In 1994, F. Chung and S.T. Yau in [2] established the following Harnack inequality for homogeneous graphs and subgraphs G with edge generating set K consisting of k generators,

$$\frac{1}{k} \sum_{a \in K} [f(x) - f(ax)]^2 + \alpha \lambda f^2(x) \leq \frac{\lambda \alpha^2}{\alpha - 2} \sup_{y \in S} f^2(y)$$

for any $\alpha > 2$ and $x \in V$, and using this Harnack inequality, they derived lower bounds for the Neumann eigenvalues and the Dirichlet eigenvalues in [2] and [3] respectively.

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In 1996, F. Chung and S.T. Yau in [5] proved the Logarithmic Harnack inequality for Ricci flat graphs, and they derived a lower bound for the log-Sobolev constant of graph G using the Logarithmic Harnack inequality.

In 2010, Y. Lin and S.T. Yau proved in [7] that the Ricci flat graphs have the non-negative Ricci curvature in the sense of Bakry and Emery. In fact, in most cases, the Ricci curvature is zero, except for some very special examples like complete graph (every two vertices of graph connected by an edge). They also proved in [7] that any locally finite connected graph satisfies either $CD(2, \frac{2}{d} - 1)$ if d is finite, or $CD(2, -1)$ if d is infinite, where $d = \sup_{x \in V} \sup_{y \sim x} \frac{d_x}{\mu_{xy}}$.

In 2014, with the assumption that the graph satisfies the curvature-dimension type inequality $CD(m, \kappa)$, F. Chung, Yong Lin and S.T. Yau in [4] established a Harnack inequality for general graphs with non-negative Ricci curvature, and they derived an eigenvalue lower bound, extending previous results for Ricci flat graphs.

In this paper, referring to the ideas in [4], with the assumption that the curvature-dimension type inequality is satisfied on graph G , we generate Logarithmic Harnack inequalities for general graphs, and get lower bounds for the log-Sobolev constant, extending the results in [5].

2. Preliminary and main theorems

Suppose G is a homogeneous graph with vertex set V and edge set E . The degree of vertex x , denoted by d_x , is the number of edges connected to x . If for every vertex x of V , d_x is finite, we say that G is a locally finite graph. The distance between two vertices is the minimum number of edges to connect them, while the diameter of G is the maximum of all the distances of the graph. We denote $x \sim y$ if vertex x is adjacent to vertex y , and μ_{xy} is the edge weight. We assume all $\mu_{xy} \equiv 1$ in this paper.

Let $V^R = \{f | f : V \rightarrow \mathbb{R}\}$, and the Laplace operator $\mathcal{L}$ of a graph G is

$$\mathcal{L}f(x) = \frac{1}{d_x} \sum_{y \sim x} [f(x) - f(y)], \quad \forall f \in V^R.$$

According to Bakry and Emery [1], we can define a bilinear operator $\Gamma : V^R \times V^R \rightarrow V^R$ by

$$\Gamma(f, g)(x) = \frac{1}{2} \{f(x)\mathcal{L}g(x) + g(x)\mathcal{L}f(x) - \mathcal{L}(f(x)g(x))\},$$

and then the Ricci curvature operator on graphs Γ_2 by iterating Γ as

$$\Gamma_2(f, g)(x) = \frac{1}{2} \{ \Gamma(f, \mathcal{L}g)(x) + \Gamma(g, \mathcal{L}f)(x) - \mathcal{L}\Gamma(f, g)(x) \}.$$

More explicitly, we have

$$\rho(x) = \frac{1}{k} \sum_{x \sim y} [f(x) - f(y)]^2,$$

$$\Gamma(f, f)(x) = \frac{1}{2} \rho(x) = \frac{1}{2} \cdot \frac{1}{k} \sum_{x \sim y} [f(x) - f(y)]^2.$$

Definition A. The operator L satisfies the curvature-dimension type inequality $CD(m, \xi)$ for some $m > 1$ if for any $f \in V^R$,

$$\Gamma_2(f, f)(x) \geq \frac{1}{m} (Lf(x))^2 + \xi \Gamma(f, f)(x).$$

We call m the dimension of the operator L and ξ the lower bound of the Ricci curvature of the operator L . It is easy to see that for $m < \tilde{m}$, the operator L satisfies $CD(\tilde{m}, \xi)$ if it satisfies $CD(m, \xi)$.

If $\Gamma_2(f, f)(x) \geq \xi \Gamma(f, f)(x)$, we say that L satisfies $CD(\infty, \xi)$.

From [4], we have the following lemma.

Lemma 1. *Suppose G is a finite connected graph satisfying $CD(m, \xi)$. Then for $x \in V$ and $f \in V^R$, we have*

$$-\frac{1}{d_x} \sum_{y \sim x} \frac{1}{d_y} \sum_{z \sim y} [f(x) - 2f(y) + f(z)]^2 \leq \left(2 - \frac{4}{m}\right) (\mathcal{L}f(x))^2 - (2 + 2\xi)\rho(x).$$

In this paper, as in [5], we define the log-Sobolev constant β of G as

$$\beta = \inf_{f \neq 0} \frac{\sum_{y \sim x} (f(x) - f(y))^2}{\sum_x f^2(x) d_x \log f^2(x)},$$

where functions f satisfy $\sum_x f^2(x) d_x = \sum_x d_x = \text{vol } G$. Then, from Theorem 5 in [5], we get the following lemma for general graphs.

Lemma 2. *For a graph G , suppose $f : V \rightarrow R$ achieves the log-Sobolev constant and $\sum_x f^2(x) d_x = \text{vol } G$. Then f satisfies, for any vertex x , $\mathcal{L}f(x) = \beta f(x) \log f^2(x)$.*

In this paper, we assume all graphs discussed are locally finite graphs with positive Ricci curvature, that is $\xi > 0$, and then by Lemma 1 and Lemma 2, we can get our main theorems as follows.

Theorem 1. *Suppose G is a graph satisfying the curvature-dimension type inequality $CD(m, \xi)$, and $f \in V^R$ satisfies $\mathcal{L}f(x) = \beta f(x) \log f^2(x)$. Then, for $U = \sup_x |f(x)| \geq 1$, the following inequality holds for $x \in V(G)$*

$$\frac{1}{d_x} \sum_{y \sim x} [f(x) - f(y)]^2 \leq \frac{2\beta^2 + 2\xi\beta}{\xi} \max \left\{ U^2 \log U^2 \left[1 + \left(1 - \frac{1}{m}\right) \log U^2 \right], 1 \right\}. \quad (1)$$

Taking $\xi = w\beta$ in inequality (1), where $w > 0$ is a constant, we get

$$\sum_{y \sim x} [f(x) - f(y)]^2 \leq \frac{2\beta d(1+w)}{w} \max \left\{ U^2 \log U^2 \left[1 + \left(1 - \frac{1}{m}\right) \log U^2 \right], 1 \right\}, \quad (2)$$

where d is the maximum degree of G .

As applications of Theorem 1, by inequality (1) and (2), we can get the following theorems.

Theorem 2. *Suppose G is a graph satisfying the curvature-dimension type inequality $CD(m, \xi)$, and $f \in V^R$ satisfies $\mathcal{L}f(x) = \beta f(x) \log f^2(x)$. Then, for $U = \sup_x |f(x)| \geq 1$, the log-Sobolev constant β of G satisfies*

$$\beta \geq \min \left(\frac{1 + 4\xi d D^2}{8d(4 - \frac{1}{m}) D^2}, \frac{1}{8D^2 d(1 + \frac{1}{w}) \log U^2 [1 + (1 - \frac{1}{m}) \log U^2]} \right),$$

where d is the maximum degree and D denotes the diameter of G .

Theorem 3. *Suppose that a graph G has isoperimetric dimension δ and satisfies the curvature-dimension type inequality $CD(m, w\beta)$. Let f denote a function achieving the log-Sobolev constant β with $\sup_x f(x) = U$. Then we have either*

$$\beta \geq \frac{w}{2d(1+w)(4c \operatorname{vol} G)^{\frac{2}{\delta}}},$$

or

$$\log U^2 \left(1 + \left(1 - \frac{1}{m} \right) \log U^2 \right) \leq 2\delta \log \delta \left[1 + \left(2 - \frac{2}{m} \right) \delta \log \delta \right].$$

Combining Theorem 2 and Theorem 3, we have

Theorem 4. Suppose that a graph G has isoperimetric dimension δ and satisfies the curvature-dimension type inequality $CD(m, w\beta)$. Let f denote a function achieving the log-Sobolev constant β with $\sup_x f(x) = U$. Then the log-Sobolev constant β of G satisfies

$$\beta \geq \min \left(\frac{c_0}{(\operatorname{vol} G)^{\frac{2}{\delta}}}, \frac{1}{16D^2 d(1 + \frac{1}{w})\delta \log \delta [1 + (2 - \frac{2}{m})\delta \log \delta]} \right),$$

where c_0 depends on δ , w , m , d , and D .

3. The proof of main theorems

By using Lemma 1 and Lemma 2, we can prove Theorem 1. The idea of proof comes from [4] and [5].

Proof of Theorem 1. We set

$$\begin{aligned} \phi(x) &= \frac{1}{d_x} \sum_{y \sim x} [f(x) - f(y)]^2 + 2\beta f^2(x) \log f^2(x) \\ &= \rho(x) + 2\beta f^2(x) \log f^2(x). \end{aligned}$$

Then, following the proof of Lemma 2.1 in [4], and by Lemma 1, Lemma 2 in Section 2, we have

$$\begin{aligned} \mathcal{L}\rho(x) &= -\frac{1}{d_x} \sum_{y \sim x} \frac{1}{d_y} \sum_{z \sim y} [f(x) - 2f(y) + f(z)]^2 \\ &\quad + \frac{2}{d_x} \sum_{y \sim x} \frac{1}{d_y} \sum_{z \sim y} [f(x) - f(y) + f(z) - f(y)] \cdot [f(x) - f(y)] \\ &\leq \left(2 - \frac{4}{m} \right) (\mathcal{L}f(x))^2 - (2 + 2\xi)\rho(x) + 2\rho(x) \\ &\quad + \frac{2}{d_x} \sum_{y \sim x} [f(x) - f(y)] \cdot \frac{1}{d_y} \sum_{z \sim y} [f(z) - f(y)] \\ &\leq \left(2 - \frac{4}{m} \right) (\mathcal{L}f(x))^2 - 2\xi\rho(x) + \frac{2}{d_x} \sum_{y \sim x} [f(x) - f(y)] \cdot (-\mathcal{L}f(y)) \\ &= \left(2 - \frac{4}{m} \right) (\mathcal{L}f(x))^2 - 2\xi\rho(x) + \frac{2}{d_x} \sum_{y \sim x} [f(x) - f(y)] \cdot (-\mathcal{L}f(y) + \mathcal{L}f(x) - \mathcal{L}f(x)) \\ &= -\frac{4}{m} (\mathcal{L}f(x))^2 - 2\xi\rho(x) + \frac{2}{d_x} \sum_{y \sim x} [f(x) - f(y)] \cdot (\mathcal{L}f(x) - \mathcal{L}f(y)) \\ &= -\frac{4}{m} (\mathcal{L}f(x))^2 - 2\xi\rho(x) \\ &\quad + \frac{2\beta}{d_x} \sum_{y \sim x} [f(x) \log f^2(x) - f(y) \log f^2(y)] [f(x) - f(y)]. \end{aligned} \tag{3}$$

On the other hand, following the proof of Theorem 6 in [5], we have

$$\begin{aligned}
& \mathcal{L}(f^2(x) \log f^2(x)) \\
&= \frac{1}{d_x} \sum_{y \sim x} [f^2(x) \log f^2(x) - f^2(y) \log f^2(y)] \\
&= \frac{2}{d_x} \sum_{y \sim x} f(x) [f(x) - f(y)] \log f^2(x) \\
&\quad - \frac{1}{d_x} \sum_{y \sim x} [f^2(x) \log f^2(x) - f^2(y) \log f^2(y)] (f(x) - f(y)) \\
&\quad + \frac{1}{d_x} \sum_{y \sim x} f(x) f(y) [\log f^2(x) - \log f^2(y)] \\
&= 2\beta f^2(x) \log^2 f^2(x) \\
&\quad - \frac{1}{d_x} \sum_{y \sim x} [f^2(x) \log f^2(x) - f^2(y) \log f^2(y)] (f(x) - f(y)) \\
&\quad + \frac{1}{d_x} \sum_{y \sim x} f(x) f(y) [\log f^2(x) - \log f^2(y)]. \tag{4}
\end{aligned}$$

By maximizing

$$\sum_{i=1}^k ab_i (\log a^2 - \log b_i^2)$$

subject to

$$\sum_{i=1}^k b_i = ka - \beta ka \log a^2,$$

where $k = d_x$, we can then use the Lagrange's method, and get that when $b_1 = b_2 = \dots = b_k$, $\sum_{i=1}^k ab_i (\log a^2 - \log b_i^2)$ attains its maximum. So, we get

$$\begin{aligned}
& \frac{1}{d_x} \sum_{y \sim x} f(x) f(y) [\log f^2(x) - \log f^2(y)] \\
&\leq \frac{1}{d_x} \sum_{y \sim x} f(x) [f(x) - \beta f(x) \log f^2(x)] [\log f^2(x) - \log (f(x) - \beta f(x) \log f^2(x))^2] \\
&\leq 2\beta f^2(x) \log f^2(x). \tag{5}
\end{aligned}$$

Thus, by (3), (4), and (5), we have

$$\begin{aligned}
& \mathcal{L}(\rho(x) + 2\beta f^2(x) \log f^2(x)) \\
&\leq -\frac{4}{m} (\mathcal{L}f(x))^2 - 2\xi \rho(x) + 4\beta^2 f^2(x) \log f^2(x) (1 + \log f^2(x)) \\
&= -\frac{4}{m} \cdot \beta^2 f^2(x) \log^2 f^2(x) - 2\xi \rho(x) + 4\beta^2 f^2(x) \log f^2(x) (1 + \log f^2(x)).
\end{aligned}$$

Now we consider a vertex v which achieves the maximum value for $\rho(x) + 2\beta f^2(x) \log f^2(x)$ over all $x \in V$, and we have

$$\begin{aligned}
0 &\leq \mathcal{L}(\rho(v) + 2\beta f^2(v) \log f^2(v)) \\
&\leq 4\beta^2 f^2(v) \log f^2(v) (1 + \log f^2(v)) - \frac{4}{m} \beta^2 f^2(v) \log^2 f^2(v) - 2\xi \rho(v) \\
&= 4\beta^2 f^2(v) \log f^2(v) + 4\beta^2 \left(1 - \frac{1}{m}\right) f^2(v) \log^2 f^2(v) - 2\xi \rho(v).
\end{aligned} \tag{6}$$

So, we have

$$\rho(v) \leq \frac{2\beta^2 f^2(v) \log f^2(v) + 2\beta^2 (1 - \frac{1}{m}) f^2(v) \log^2 f^2(v)}{\xi}.$$

Therefore for every $x \in V$, we have

$$\begin{aligned}
&\frac{1}{d_x} \sum_{y \sim x} [f(x) - f(y)]^2 + 2\beta f^2(x) \log f^2(x) \\
&\leq \rho(v) + 2\beta f^2(v) \log f^2(v) \\
&= \frac{2\beta^2 + 2\xi\beta}{\xi} f^2(v) \log f^2(v) \left[1 + \frac{\beta}{\beta + \xi} \cdot \left(1 - \frac{1}{m}\right) \log f^2(v)\right].
\end{aligned} \tag{7}$$

Since $m > 1$, $\xi > 0$, and $\beta > 0$, for $U = \sup_x |f(x)| \geq 1$, we have

$$\frac{1}{d_x} \sum_{y \sim x} [f(x) - f(y)]^2 \leq \frac{2\beta^2 + 2\xi\beta}{\xi} \max \left\{ U^2 \log U^2 \left[1 + \left(1 - \frac{1}{m}\right) \log U^2\right], 1 \right\}.$$

This completes the proof. $\square$

The proof of [Theorem 2](#) is very similar to Theorem 7 in [\[5\]](#). We give the proof simply for the readers' convenience.

Proof of Theorem 2. We consider two cases:

1. $\sup_x |f(x) - 1| \leq \frac{1}{2}$. From [\[4\]](#), we have $\lambda_1 \geq \frac{1+4\xi dD^2}{d \cdot (8 - \frac{2}{m} \cdot D^2)}$. So we get

$$\beta \geq \frac{\lambda_1}{4} \geq \frac{1 + 4\xi dD^2}{8d \cdot (4 - \frac{1}{m} \cdot D^2)}.$$

2. $\sup_x |f(x) - 1| \geq \frac{1}{2}$. Since $\sum_v f^2(v) d_x = \text{vol } G$, we can take $f(x_0) = c$ and $f(x_s) = 1$. Let P be a shortest path with vertices $x_0, \dots, x_s$, where $f(x_0) = \sup_x f(x)$. When $U^2 \log U^2 [1 + (1 - \frac{1}{m}) \log U^2] \geq 1$, since $s \leq D$, by [\(2\)](#) we have

$$\sum_{i=0}^{s-1} (f(x_i) - f(x_{i+1}))^2 \leq \frac{2\beta dD(1+w)}{w} \cdot U^2 \log U^2 \left[1 + \left(1 - \frac{1}{m}\right) \log U^2\right].$$

On the other hand,

$$\sum_{i=0}^{s-1} (f(x_i) - f(x_{i+1}))^2 \geq \frac{U^2}{4D}.$$

So we get

$$\beta \geq \frac{1}{8D^2d(1 + \frac{1}{w}) \log U^2[1 + (1 - \frac{1}{m}) \log U^2]}.$$

When $U^2 \log U^2[1 + (1 - \frac{1}{m}) \log U^2] \leq 1$, we can easily get

$$\beta \geq \frac{1}{8D^2d(1 + \frac{1}{w})}.$$

This completes the proof. $\square$

The proof of Theorem 3 is very similar to Theorem 8 in [5]. We also give the proof simply here.

Proof of Theorem 3. We take P as a shortest path with vertices $x_0, \dots, x_s$, where $f(x_0) = U$ and $f(x_s) \leq 1$. To upper bound U , we consider the ball $B_r = \{y : d(x_0, y) \leq r\}$. We take

$$s = \left\lfloor \frac{1}{2\sqrt{\frac{2\beta d(1+w)}{w}} \log U^2[1 + (1 - \frac{1}{m}) \log U^2]} \right\rfloor.$$

By (21) in [5], we have

$$\beta^{\frac{\delta}{2}} \text{vol } G \geq \frac{U^2}{4c[\frac{2d(1+w)}{w} \log U^2(1 + (1 - \frac{1}{m}) \log U^2)]^{\frac{\delta}{2}}}.$$

We consider two cases:

1. when $\frac{U^2}{[\log U^2(1 + (1 - \frac{1}{m}) \log U^2)]^{\frac{\delta}{2}}} \geq 1$, we have

$$\beta \geq \frac{w}{2d(1+w)(4c \text{vol } G)^{\frac{2}{\delta}}}.$$

2. when $\frac{U^2}{[\log U^2(1 + (1 - \frac{1}{m}) \log U^2)]^{\frac{\delta}{2}}} < 1$, that is,

$$\log U^2 \left(1 + \left(1 - \frac{1}{m} \right) \log U^2 \right) \leq 2\delta \log \delta \left[1 + \left(2 - \frac{2}{m} \right) \delta \log \delta \right].$$

So we complete the proof. $\square$

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